

## The Nature of Statistical Fluctuations with Applications to Cosmic Rays

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I. The square of the mean deviation  $D$  of the combined effect of several random processes releasing an average of  $x, y, z \dots$  particles per unit time and producing  $a, b, c \dots$  ion pairs, respectively, per particle, is  $D^2 = a^2x + b^2y + c^2z + \dots$ , regardless of whether the separate effects are added or subtracted by the experimental arrangement. For tube-counters, point-counters, scintillation screens and particle counting chambers,  $a=b=c=1$ ; for ionization chambers  $a, b, c \dots$  are unequal. II. From the standpoint of statistical fluctuations, the use of two identical instruments in a differential circuit is inferior to the use of a single instrument. III. The natural observational limit for the measurement of  $x$  particles against a background of  $y$  particles is  $x=0.67(y)^{1/2}$ . IV. The statistical fluctuations in the ionization produced by cosmic rays in a spherical ionization chamber are treated rigorously and the fluctua-

tions due to heterogeneity of range and to showers are derived. V. Application to existing data shows that the showers observed in cloud-chamber photographs of the cosmic radiation are also present in the ionization chamber in about the same frequency and multiplicity as indicated by the cloud-chamber results. The tube-counter investigations of the cosmic-ray flux are also in agreement with the deductions from the statistical fluctuations in the ionization chamber. An upper limit of  $70 \pm 10$  ion pairs per cm in air at 1 atmosphere is set for the *total* ionization along the path of an individual cosmic-ray secondary. The size and the relative frequency of occurrence of showers is appreciably greater at 14,700 feet elevation than at sea level. These showers are quite distinct from the ionization bursts or Stösse observed by Hoffmann, Steinke and others.

### INTRODUCTION

MANY physical processes which appear to proceed at a continuous and uniform rate are shown to be discontinuous and spasmodic when examined with very sensitive apparatus. Thus the emission of  $\alpha$ -particles from radioactive bodies has long been known to consist of the emission of individual particles randomly distributed in time. Ionization by x-rays, disintegration of matter by  $\alpha$ -particles, protons, deuterons or neutrons, and the flux of cosmic-ray secondaries are similar processes which exhibit an apparently constant rate when large effects are measured over long time intervals, but show measurable fluctuations from the average rate when feeble sources or short time intervals are employed.

The mathematical theory of these fluctuations for the simple case of  $\alpha$ -particle emission has been given by H. Bateman<sup>1</sup> and others, and dis-

continuities in light emission and x-ray absorption have been considered by N. Campbell<sup>2</sup> and by Colm-Peters and Lange.<sup>3</sup> The theory of statistical fluctuations is here generalized, after which several modern physical problems are treated as special cases of the general theory.

### THE GENERAL EQUATION

We shall assume that the physical process involved takes place at an average rate which is constant. This is true for the flux of cosmic-ray particles and for many other processes. If the emission of particles from radioactive bodies is considered, the decay of the parent substance must be negligible during the time of observation, otherwise the probability of emission and the average rate of emission would not remain constant. If  $\lambda$  is the average rate of appearance of particles in the measuring instrument and  $t$  is the length of the unit of time over which the individual observations are made, then  $\lambda t$  is the average number of particles measured in a single

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<sup>1</sup> H. Bateman, *Phil. Mag.* **20**, 704 (1910). E. v. Schweidler, *Premier Congrès intern. de Radiologie*, Liege 1905. K. W. F. Kohlrausch, *Ergebnisse der exakten Naturwissenschaften* **5**, 192 (1926).

<sup>2</sup> N. Campbell, *Proc. Camb. Phil. Soc.* **15**, 301 (1909).

<sup>3</sup> Colm-Peters and Lange, *Ann. d. Physik* **4**, 453 (1930).

observation. Call this average value  $x$ . Then the probability,  $P_l$ , that  $l$  particles, instead of  $x$  particles, will appear in a single observation period is

$$P_l = x^l e^{-x} / l! , \quad (1)$$

as is proved in detail by Bateman.<sup>1</sup>

If each particle produces a specific effect,  $a$ , (e.g.,  $a$  ion pairs per particle) then the variation,  $v_l$ , from the average value  $ax$  is  $a(l-x)$  when  $l$  particles appear. Employing the definition of Bessel and Gauss, we speak of the mean deviation,  $D$ , which is the square root of the most probable value of the square of the deviation. The most probable value being the arithmetic average of a large number,  $n$ , of observations, we have

$$D^2 = \sum v_l^2 / n. \quad (2)$$

In the general experimental case, there will be several processes acting simultaneously on the measuring apparatus. Thus, in ionization chamber measurements of  $\alpha$ -particles, there will also be present a background ionization due to cosmic-ray secondaries and to  $\alpha$ -particles emitted from radioactive contamination in the walls of the apparatus. The statistical fluctuations observed in the apparatus will therefore be due to the joint action of the fluctuations from several independent and individually random processes. Let these independent processes produce an average of  $x, y, z, \dots$ , particles, respectively, in the unit of time chosen. Let each particle from the first process produce a specific effect  $a$  and each particle from the other processes specific effects  $b, c, \dots$ , respectively, measured in the same units. In any single observation each process will produce  $l, m, n, \dots$ , particles, instead of the average value  $x, y, z, \dots$ , particles. Then the net variation due to the joint action of the separate processes will be  $a(l-x) + b(m-y) + c(n-z) + \dots$ , if the effects from the separate processes are additive, as they are in the single ionization chamber experiment considered above. The probability that  $l, m, n, \dots$ , particles will be produced by the several independent processes is  $P_l P_m P_n \dots = (x^l y^m z^n \dots / l! m! n! \dots) e^{-x-y-z-\dots}$ , hence the mean deviation is given by

$$D^2 = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \dots \frac{x^l y^m z^n \dots}{l! m! n! \dots} e^{-x-y-z-\dots} \times [a(l-x) + b(m-y) + c(n-z) + \dots]^2. \quad (3)$$

Upon expansion, Eq. (3) reduces to

$$D^2 = a^2 x + b^2 y + c^2 z + \dots, \quad (4)$$

which is the general equation for the statistical fluctuations due to the joint action of any number of independent processes.

If the apparatus arrangement involves the instrumental subtraction of one effect from another, as in differential ionization chamber measurements, the signs between the parentheses in the final bracket of Eq. (3) change but Eq. (4) remains unaltered. Eq. (4) is therefore general for any number of processes and for any arrangement of the apparatus.

In any finite series of observations we can never actually measure the exact average rate of appearance of particles. We can only find its most probable value, which is the arithmetic average obtained from a large number  $n$  of independent observations. The probable variation  $r$  of a single observation, i.e., that variation which is equally as likely to be exceeded by the observed variation  $v$  as not, is

$$r = 0.6745 (\sum v^2 / (n-1))^{1/2}. \quad (5)$$

Combining Eqs. (2) and (5) we find  $D$  related to  $r$  by

$$r = 0.6745 D (n / (n-1))^{1/2}. \quad (6)$$

The summation involved in Eq. (3) is equivalent to making an infinite number of observations, hence when applied to Eq. (4), Eq. (6) becomes

$$r = 0.6745 D. \quad (7)$$

The Poisson law of Eq. (1) gives an asymmetric distribution about the average value  $x$ , favoring values of  $l$  which are less than  $x$ . The asymmetry is the order of 10 to 5 percent for  $x = 10$  to 100 and vanishes as  $x \rightarrow \infty$ , when the distribution becomes essentially Gaussian. The asymmetry for  $x \geq 10$  does not invalidate Eq. (7).

## DIFFERENTIAL CIRCUITS

In the detection of minute quantities of radioactive substances with ionization chambers,<sup>4</sup> the detection of mitogenetic rays with tube-counters<sup>5</sup> and in many similar problems, observers have used two instruments in a differential hook-up, thus cancelling out the cosmic-ray background by means of a dummy chamber or counter. It has been claimed by some experimenters that a differential circuit also cancels out the statistical fluctuations in the background, the magnitude of which usually imposes the limit of precision on the measurements. Actually, when the background is received independently by two identical instruments, the resulting statistical variation is  $2^{\frac{1}{2}}$  times the variation for a single instrument, regardless of whether the two instruments are connected to oppose or to assist each other. This can be seen from Eq. (4), where, for strictly differential circuits,  $a=b$  and  $x=y$  and these refer to the background in the two identical instruments, while  $c$  and  $z$  refer to the effect being studied.

Whether a single instrument or a differential circuit employing two such instruments is used, a control, or background test, must be made, independent of the actual run which consists of the background plus the effect being measured. In taking the difference between these two readings as a measure of the effect, the statistical fluctuation in the background enters both measurements. Since the statistical fluctuation of the background is  $2^{\frac{1}{2}}$  times larger in the differential circuit than in the single circuit, the uncertainty of measurement, which arises from these fluctuations, is greater when the differential circuit is used. Thus if the average background in a single instrument is  $x$  particles during the time interval employed, the value of the background actually observed will be  $x \pm 0.67 (x)^{\frac{1}{2}}$ , by Eqs. (4) and (7). Similarly, the background in a differential circuit with two such instruments will be  $0 \pm 0.67 (2x)^{\frac{1}{2}}$ . When an effect to be measured, consisting of  $y$  particles, is added, the reading on the single instrument will be  $x+y \pm 0.67 (x+y)^{\frac{1}{2}}$ , and on the

differential instrument  $y \pm 0.67 (2x+y)^{\frac{1}{2}}$ . Now in subtracting the background from these readings, in order to obtain the value of the effect  $y$ , we have for the single instrument,

$$[x+y \pm 0.67 (x+y)^{\frac{1}{2}}] - [x \pm 0.67 (x)^{\frac{1}{2}}] \\ = y \pm 0.67 (2x+y)^{\frac{1}{2}},$$

and for the differential instrument

$$[y \pm 0.67 (2x+y)^{\frac{1}{2}}] - [0 \pm 0.67 (2x)^{\frac{1}{2}}] \\ = y \pm 0.67 (4x+y)^{\frac{1}{2}},$$

by the principles of the propagation of errors. Thus the uncertainty is greater in the differential circuit by an amount which depends on the ratio of  $y$  to  $x$ . If  $y \gg x$ , the two apparatus arrangements have equal merits but if  $y \ll x$ , as is the case near the observational limit, then the fluctuations are  $2^{\frac{1}{2}}$  times larger in the differential circuit. Differential circuits, though suffering from this defect, do possess advantages which at times outweigh this disadvantage. Their effectiveness in spreading a differential effect over a large scale, thus permitting more accurate readings, and in balancing out battery variations, transient local radiations and other systematic errors often justifies their use, even near the natural observational limit.

## THE NATURAL OBSERVATIONAL LIMIT

Many physical measurements involve the detection of some effect over and above a background or "zero effect." When the effect to be measured becomes so small that it is just equal to the probable statistical variation in the background the observer cannot establish its presence and the natural observational limit is reached.<sup>6</sup> If the small effect being observed is the emission of  $x$  particles per unit time, and each particle produces a specific effect  $a$  (e.g.,  $a$  ion pairs per particle) while the background is due to  $y$  particles of specific effect,  $b$ ,  $z$  particles of specific effect,  $c$ , etc., then the observational limit is defined by

$$r = ax = 0.67 D = 0.67 (a^2x + b^2y + c^2z + \dots)^{\frac{1}{2}} \quad (8)$$

and if\*  $a^2 \ll 4(b^2y + c^2z + \dots)/(0.67)^2$ , the observational limit is

<sup>4</sup> Halledauer, Akad. Wiss., Wien **134**, 39 (1925); Paneth and Koech, Zeits. f. physik. Chemie, Bodenstein-Festband, 145 (1931); Evans, Rev. Sci. Inst. **4**, 223 (1933).

<sup>5</sup> Siebert and Seffert, Naturwiss. **21**, 193 (1933).

<sup>6</sup> Evans, Rev. Sci. Inst. **4**, 229 (1933).

\* This means also that  $4x \gg 1$ .

$$ax = 0.67 (b^2y + c^2z + \dots)^{\frac{1}{2}}. \quad (9)$$

Eq. (9) applies to all ionization and counting measurements. When tube-counters, point-counters, scintillation screens or particle-counting chambers are used,  $a = b = c = 1$ .

The common case of counting  $x$  particles per unit time against a background of  $y$  particles per unit time has the observational limit  $x = 0.67 (y)^{\frac{1}{2}}$ . Substituting appropriate values for  $x$ , we find, for example, in  $\alpha$ -ray emanation-measurements of minute quantities of radon, that if  $y$  is the number of  $\alpha$ -particles emitted per hour from the walls of the ionization chamber, then the natural observational limit in grams of radium or curies of radon is  $0.36 \times 10^{-14} (y)^{\frac{1}{2}}$  for a one hour reading. Since an ordinary brass cylindrical ionization chamber 15 cm high and 15 cm in diameter emits about 200  $\alpha$ -particles per hour into the chamber, such a typical chamber has a practical natural observational limit of  $5.1 \times 10^{-14}$  g Ra. If  $y$  is reduced to 50, one-fourth of its usual value, the limit is  $2.5 \times 10^{-14}$  g Ra for a one hour reading. The radon and decay products from  $10^{-14}$  g Ra send 2.5  $\alpha$ -particles per hour into an ionization chamber. It is therefore difficult to substantiate the suggestion that  $10^{-14}$  g Ra can be measured with certainty in a series of ten 7 minute readings, as has been asserted by Halledauer.<sup>4</sup>

Precision in measurements approaching the natural observational limit imposed by the statistical variations of the background can be improved only by lengthening the time of observation or by decreasing the absolute magnitude of the background. Methods for doing this in ionization chambers have been described elsewhere.<sup>6</sup> Lead shields may be used to absorb local  $\gamma$ -rays, electrometers may be evacuated, lamp-black may be painted on the inside wall of the ionization chamber to absorb wall  $\alpha$ -rays or a Hoffmann net may be used for the same purpose.

In the use of tube-counters there is another method available. This consists of protecting the counter on which the active agent acts, by surrounding it with other counters which will respond to coincidences between the central counter and these guard counters. In this way the background of long range cosmic-ray secondaries in the central counter may be cancelled exactly, particle by particle, and the observational limit

for any feeble effect which is superposed on the background in the central counter may be greatly decreased.  $\gamma$ - or x-ray background should be eliminated by absorption in lead which is free from its isotope, Ra D. Either a pentode or triode vacuum tube amplifiers connected in a  $Y$  circuit may be used, the output being arranged to record only those events in the central counter which are not coincident with counts in the outside guard counters. Thus a cosmic-ray secondary entering one of the guard counters, or passing through both a guard counter and the central counter, will not be recorded. Since nearly all the cosmic-ray secondaries traversing the central counter will have originated outside it and will, therefore, have traversed a guard counter, nearly all the background due to cosmic rays in the central counter can be eliminated.

#### COSMIC RAYS IN SPHERICAL IONIZATION CHAMBERS

Cosmic-ray secondaries traversing an ionization chamber have various effective path lengths, depending upon what part of the chamber they traverse. This heterogeneity of effective range introduces a term in the equations describing the fluctuations. Since the ionization  $j$ , per cm of path, is nearly the same for all the secondaries, the fluctuations due to heterogeneity of range may be computed from Eq. (4) and the geometry of the ionization chamber. Eq. (4) also will describe the fluctuations due to the occurrence of showers of associated secondaries in the ionization chamber.

For purposes of geometrical analysis, we consider only particles moving in one direction. Where the total number of particles is large, this involves no loss in generality. If there are  $n_1$  particles having a track length  $h_1$  and hence a total ionization  $A_1 (= h_1 j)$  and  $n_2$  particles with ionization  $A_2$ , etc., then the total ionization  $Q$ , and the average value of the square of the mean deviation,  $D^2 \equiv (\delta Q)^2$ , are given by:

$$Q = A_1 n_1 + A_2 n_2 + \dots \equiv A \sum n = A (N \pi R^2), \quad (10)$$

$$\begin{aligned} (\delta Q)^2 &= A_1^2 n_1 + A_2^2 n_2 + \dots \\ &\equiv \omega A^2 \sum n = \omega A^2 (N \pi R^2), \quad (11) \end{aligned}$$

where  $A$  is the average number of ion pairs per particle,  $N$  is the number of particles per  $\text{cm}^2$  in the time interval  $t$ ,  $R$  is the radius of the ionization chamber and  $\omega$  is a numerical coefficient which we shall now evaluate for the case of the spherical ionization chamber. In Fig. 1 consider

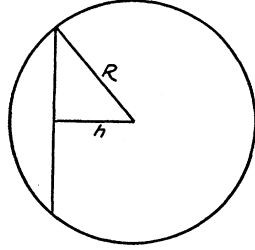


FIG. 1.

the particles which pass vertically through the spherical chamber;  $h$  is the perpendicular distance from the path of such a particle to the center of the chamber. The length of the path of such a particle is  $2(R^2 - h^2)^{1/2}$ , and its ionization is  $2j(R^2 - h^2)^{1/2}$ , while there are  $N2\pi h dh$  such particles between  $h$  and  $h + dh$  from the center. Eqs. (10) and (11) then become:

$$Q = \int_0^R 2j(R^2 - h^2)^{1/2} N 2\pi h dh = 4\pi j N R^3 / 3, \quad (12)$$

$$(\delta Q)^2 = \int_0^R 4j^2(R^2 - h^2) N 2\pi h dh = 2\pi j^2 N R^4. \quad (13)$$

Combining these results with the final terms of Eqs. (10) and (11) we find, for spherical ionization chambers:

$$A = 4jR/3, \quad (14)$$

$$\omega = 9/8. \quad (15)$$

Cloud-chamber photographs and the new theories of electron pair production have disclosed the association of double, triple and higher multiple groups of cosmic-ray tracks in showers. If  $x_1$  is the total number of single cosmic-ray secondaries traversing the chamber in the unit time interval  $t$ , and  $a_1$  the average total ionization per track ( $= 4jR/3$ ),  $x_2$  the number of associated pairs of tracks, and  $a_2$  the ionization produced by the average pair ( $= 2a_1$ ), while  $x_3, \dots, x_n$  and  $a_3, \dots, a_n$  are the analogous quantities for higher multiple tracks up to showers of  $n$  tracks, then

$$\begin{aligned} Q &= a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n \\ &= a_1 x_1 \left( 1 + \frac{a_2 x_2}{a_1 x_1} + \frac{a_3 x_3}{a_1 x_1} + \dots + \frac{a_n x_n}{a_1 x_1} \right) \\ &= a_1 x_1 (1 + 2r_2 + 3r_3 + \dots + nr_n) \\ &\equiv a_1 x_1 \sum n r_n, \end{aligned} \quad (16)$$

$$\begin{aligned} (\delta Q)^2 &= \omega (a_1^2 x_1 + a_2^2 x_2 + a_3^2 x_3 + \dots + a_n^2 x_n) \\ &= (9/8) a_1^2 x_1 (1 + 4r_2 + 9r_3 + \dots + n^2 r_n) \\ &\equiv (9/8) a_1^2 x_1 \sum n^2 r_n, \end{aligned} \quad (17)$$

where  $r_n \equiv x_n/x_1$ , and is the relative frequency of occurrence of showers of  $n$  tracks with respect to single tracks. Combining Eqs. (14), (16) and (17), we obtain

$$j \cdot \sum n^2 r_n / \sum n r_n = (2/3R) \cdot (\delta Q)^2 / Q. \quad (18)$$

If  $N_1$  is the number of single tracks per  $\text{cm}^2$  per unit time  $t$ , an ideal tube counter, which would record each single particle and each shower of  $n$  particles as one count each, would register  $N$  counts per  $\text{cm}^2$  per unit time, where

$$N = N_1 \sum r_n = x_1 \sum r_n / \pi R^2. \quad (19)$$

Combining Eqs. (14), (16) and (19), we obtain

$$j \cdot N \cdot \sum n r_n / \sum r_n = 3Q/4\pi R^3. \quad (20)$$

The terms  $\sum n^2 r_n / \sum n r_n$  and  $\sum n r_n / \sum r_n$  occurring in the final Eqs. (18) and (20) are directly determined by the frequency of occurrence  $r_n$  and the multiplicity  $n$  of the showers of associated cosmic-ray secondaries such as would be observed in a cloud chamber having the same geometry as the ionization chamber. The summations  $\sum n^2 r_n$ ,  $\sum n r_n$  and  $\sum r_n$  and their ratios in Eqs. (18) and (20) are each unity when no multiple tracks exist and each exceeds unity when showers are present,  $\sum n^2 r_n / \sum n r_n$  increasing more rapidly than  $\sum n r_n / \sum r_n$ .

#### COMPARISON WITH AVAILABLE COSMIC-RAY DATA

Eqs. (18) and (20) permit a direct test of the consistency of measurements made with the spherical ionization chamber, cloud chamber and tube-counter. The recent discovery by Anderson of the positive electron and of the production by cosmic radiation of positive and negative elec-

trons in pairs suggests that many, if not all, of the single cosmic-ray tracks observed in cloud-chamber photographs are in reality isolated branches of showers occurring near the apparatus. The recent observations of Rossi and others indicate that elements of high atomic number are more efficient in the production of large showers than those of low atomic number. Therefore, for the rigorous application of Eqs. (18) and (20) the geometry and shielding of the ionization chamber, cloud chamber and tube-counter should be identical. No such apparatus exist, but, bearing in mind these limitations, the application of Eqs. (18) and (20) to existing data does yield preliminary results of some interest.

The frequency  $r_n$  and multiplicity  $n$  of showers in cylindrical cloud chambers have been studied by several observers. Quantitative agreement is poor because the large showers are quite rare and enough photographs to give a good statistical average have not yet been taken. Where the cloud chamber is expanded by a tube-counter coincidence arrangement, as in Blackett and Occhialini's<sup>7</sup> apparatus and in a recent modification of the Anderson-Millikan arrangement for the measurement of particle energies up to  $3 \times 10^9$  volts, the natural distribution of multiple showers is distorted, since even a small shower in the apparatus has a much greater chance of setting off the coincident counters than has an isolated single track. The number of photographs of tracks and the corresponding values of the summation ratios for Eqs. (18) and (20), computed from the occurrence of multiple tracks in randomly expanded cloud chambers, are given in Table I.

TABLE I. *Distribution of showers in cloud-chamber observations.*

$n$	Number of Photographs							$\frac{\sum n^2 r_n}{\sum n r_n}$	$\frac{\sum n r_n}{\sum r_n}$
	1	2	3	4	5	6-10	over 10		
Anderson <sup>8</sup>	708	82	7	7	3	7	1	1.92	1.23
Locher <sup>9</sup>	148	14	5	0	1	0	0	1.40	1.17
Skobelzyn <sup>10</sup>	23	3	1	0	0	0	0	1.51	1.15

<sup>7</sup> Blackett and Occhialini, Proc. Roy. Soc. **A139**, 699 (1933).

<sup>8</sup> Anderson, Phys. Rev. **44**, 406 (1933).

<sup>9</sup> Locher, Phys. Rev. **39**, 883 (1932).

<sup>10</sup> Skobelzyn, Zeits. f. Physik **54**, 686 (1929).

The number of counts per  $\text{cm}^2$  per minute in a tube-counter near sea level is rather uncertain because of the difficulty of eliminating local  $\gamma$ -rays and radioactive contamination in the counter but it is near unity. Blackett and Occhialini<sup>7</sup> suggest 1.5 counts per  $\text{cm}^2$  per minute, Bothe and Kölhorster<sup>11</sup> found about 1.0, Kölhorster and Tuwim<sup>12</sup> and Mott-Smith and Locher<sup>13</sup> found less than 1.0, Korff<sup>14</sup> finds approximately 1 in Pasadena. Values for the ionization  $j$  in ion pairs per cm of path in air at one atmosphere given by various observers are as follows: Skobelzyn,<sup>10</sup> 40 ion pairs per cm; Bothe and Kölhorster,<sup>11</sup> 90; Kölhorster and Tuwim,<sup>12</sup> 135; Messerschmidt,<sup>15</sup> 110; Locher,<sup>16</sup> 36; Blackett and Occhialini,<sup>7</sup> 80; Anderson,<sup>8</sup> 31 for primary ionization by drop-counts on diffuse tracks, and 120 to 140 from energy loss in lead. Theoretical considerations by Carlson and Oppenheimer<sup>17</sup> and by Heisenberg<sup>18</sup> suggest that the primary ionization by high energy electrons, such as cosmic-ray secondaries, should not exceed 50 to 60 ion pairs per cm in standard air unless the energy of the electron is far above  $10^9$  electron-volts.

The continuous photographic records of cosmic-ray intensity now being obtained by R. A. Millikan and H. V. Neher provide more suitable data for application to Eqs. (18) and (20) than the data of Messerschmidt<sup>15</sup> because of the higher charge sensitivity of their electroscopes. The records are conveniently divided into 15 minute intervals and these exhibit statistical variations of the order of 2 percent at sea level. Series of 66 quarter hours at Pasadena and of 55 quarter hours at 14,700 feet in the Peruvian Andes have been analyzed as typical examples of the present method. The electroscope is 15 cm in diameter, contains air at 30 atmospheres and is shielded from local radiation by 4 inches of lead. The observed ionization  $Q$  and the statistical variations  $(\delta Q)^2$  about the mean value were corrected for lack of saturation of the ion-current, for ionization

<sup>11</sup> Bothe and Kölhorster, Zeits. f. Physik **56**, 751 (1929).

<sup>12</sup> Kölhorster and Tuwim, Zeits. f. Physik **73**, 130 (1931).

<sup>13</sup> Mott-Smith and Locher, Phys. Rev. **38**, 1399 (1931).

<sup>14</sup> Korff, personal communication, June 1933.

<sup>15</sup> Messerschmidt, Zeits. f. Physik **78**, 668 (1932).

<sup>16</sup> Locher, Phys. Rev. **39**, 883 (1932).

<sup>17</sup> Carlson and Oppenheimer, Phys. Rev. **41**, 763 (1932).

<sup>18</sup> Heisenberg, Ann. d. Physik **13**, 430 (1932).

bursts or Stösse, for the electroscope's "zero," due principally to  $\alpha$ -particles from the walls, and for the observational uncertainty in reading the photographs. The uncertainty in each of these corrections is small compared with the statistical variations in cosmic-ray intensity. The usual barometer and temperature corrections were made; these exert no influence on the statistical variations. Comparison of several short series of observations shows that  $(\delta Q)^2$  can be obtained to within  $\pm 15$  percent, which is better than the present agreement in the various values of  $j$ ,  $N$  and  $\sum n^2 r_n / \sum n r_n$ , as shown above. Tables II

TABLE II. *The distribution of showers in the electroscope.*

	$\frac{(\delta Q)^2}{Q}$	$j \cdot \frac{\sum n^2 r_n}{\sum n r_n}$	$\frac{\sum n^2 r_n}{\sum n r_n}$			
	at 30 atm.	at 1 atm.	$j=50$	$j=80$	$j=100$	$j=150$
Pasadena	$4.38 \times 10^4$	130.	2.60	1.63	1.30	0.87
14,700 ft. elev.	$7.75 \times 10^4$	230.	4.60	2.88	2.30	1.53

TABLE III. *The flux of cosmic-ray secondaries.*

	$Q$	$N$			
	at 30 atm.	$j=50$	$j=80$	$j=100$	$j=150$
Pasadena	$8.43 \times 10^7$	1.7	1.1	0.86	0.57
14,700 ft. elev.	$20.8 \times 10^7$	4.2	2.6	2.1	1.4

and III contain the results of the analysis of the two sets of data mentioned.  $Q$  and  $(\delta Q)^2/Q$  are in terms of total ion pairs formed in the electroscope per 15 minutes in air at 30 atmospheres;  $j$  is ion pairs per cm in air at 1 atmosphere;  $N$  is effective counts per  $\text{cm}^2$  per minute, as defined by Eq. (19).

Eq. (18) involves two unknowns,  $j$  and  $\sum n^2 r_n / \sum n r_n$  and only their product can be evaluated from the electroscope data. Table II therefore shows the distribution of showers in the electroscope when  $j$  is given values ranging from 50 to 150 ion pairs per cm. These shower values may be compared with Anderson's Pasadena values shown in Table I, while remembering that his cylindrical cloud chamber is 16.5 cm in diameter and 4 cm deep and is shielded<sup>8</sup> principally by copper and iron, whereas Millikan and Neher's spherical electroscope is 15 cm in diameter and is shielded by 10 cm lead. Because of shielding and

geometrical differences it is therefore to be expected that showers would be more abundant in the electroscope. The data therefore dictate an upper limit of 70 ion pairs for the *total* ionization per cm along a cosmic-ray secondary track in air at 1 atmosphere and suggest that the actual value is appreciably lower, a conclusion which agrees with Anderson's<sup>8</sup> latest observations and with theoretical considerations.<sup>17, 18</sup>

The large statistical fluctuations in the electroscope at 14,700 feet elevation demand about a twofold increase in the shower abundance expression,  $\sum n^2 r_n / \sum n r_n$ , assuming that  $j$  is not appreciably changed. The softer components of the cosmic radiation, present at high altitudes, therefore appear to produce considerably larger showers of associated tracks than do the harder, sea-level components.

Eq. (20) involves three unknowns,  $j$ ,  $N$ , and  $\sum n r_n / \sum r_n$ , their product being given by the electroscope data. Table III shows the values for  $N$ , the number of counts per  $\text{cm}^2$  per minute, which are obtained by assuming Anderson's value for  $\sum n r_n / \sum r_n$  from Table I and assigning values from 50 to 150 ion pairs per cm to  $j$ . Here again, because of geometry and shielding, the cloud-chamber value of the shower distribution term,  $\sum n r_n / \sum r_n$ , is undoubtedly lower than the electroscope's value and hence the upper limit of the product  $j \cdot N$  is given by Table III. Comparison of the Pasadena values with the published results on  $N$  for tube-counters, summarized above, shows an upper limit of about 80 ion pairs per cm for  $j$  and suggests a somewhat lower value.

At 14,700 feet elevation the increased abundance of large showers, shown by Table II, requires an increase of the shower distribution term,  $\sum n r_n / \sum r_n$ , of Eq. (20). Again assuming  $j$  to be independent of altitude, Eq. (20) shows that the counting rate  $N$  for tube-counters does not rise as rapidly with increasing altitude as does the ionization  $Q$  in the electroscope. This is because the tube-counter fails to distinguish between showers and single particles.

The high and low altitude showers here considered produce much less total ionization and are entirely distinct from the ionization bursts or Stösse reported by Hoffmann, Steinke and many other observers.

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